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SOME STABILITY CONDITIONS FOR A COMPRESSIBLE ELASTIC MATERIAL

K. N. SAWYERS and R. S. RIVLIN

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Some Stability Conditions for a Compressible Elastic Material

by

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Abstract

Two sets of restrictions on the strain-energy function for a compressible isotropic elastic material are obtained which are necessary conditions for stability of the material. These arise from the following considerations. (i) A rectangular block is subjected to a finite pure homogeneous deformation and an infinitesimal pure homogeneous deformation with arbitrary principal directions is superposed. The dimensions in two of these principal directions are held constant. Then the incremental modulus associated with the third principal direction must be positive for stability to obtain. (ii) In the initial pure homogeneous deformation one pair of faces of the block is force-free. The superposed infinitesimal pure homogeneous deformation has one of its principal directions normal to these faces, which remain force-free, and the principal extension ratio corresponding to another is unity. The incremental modulus corresponding to the third principal direction must be positive for stability to obtain.

1. Introduction

It has been argued elsewhere [1] that although convexity of the strain-energy function with respect to arbitrary perturbations from an equilibrium state of finite deformation is not a necessary condition for stability of an elastic material, convexity with respect to certain restricted classes of perturbations is necessary.

In earlier papers the restrictions imposed on the strain-energy function for an isotropic elastic material by certain considerations of this type have been obtained. Baker and Ericksen [2] and Ericksen [3] obtained the restrictions on the strain-energy function which - although it was not presented in this way - express the condition that the incremental shear modulus in an isotropic elastic material, compressible or incompressible, be positive for simple shears superposed on a pure homogeneous deformation, when the direction of shear is a principal direction and the plane of shear is a principal plane. Sawyers and Rivlin [4,5] obtained corresponding conditions in the case when the material is incompressible and the plane of shear is a principal plane, but the direction of shear is arbitrary in that plane. In [4], as in [3], the result was obtained in the form of the equivalent conditions that the secular equation for the determination of the velocities of plane waves of infinitesimal amplitude propagated in a principal plane of the underlying pure homogeneous deformation yield only real solutions for all directions of propagation in that plane. In [6] the restriction in [4] of the direction of propagation to a principal plane is removed and in [7] the restriction of the plane of shear of the superposed infinitesimal shear to a principal plane is removed. However, the results obtained in [6] and [7] are in a less explicit form than those in [4] and [5].

In [8] conditions that the secular equation for the determination of the velocities of plane waves of infinitesimal amplitude, propagated in a principal plane of the underlying pure homogeneous deformation, have only real solutions are obtained in the case when the isotropic elastic material is compressible. Unlike the situation when the material is incompressible, these conditions are not equivalent to those that the incremental shear modulus be positive, since the waves are not necessarily transverse shear waves.

In the present paper two further sets of necessary conditions for material stability are obtained for the case when the material is compressible. One of these arises from the following consideration. A rectangular block of the isotropic elastic material is subjected to a finite pure homogeneous deformation and then an infinitesimal pure homogeneous deformation with two of its principal extension ratios equal to unity is superposed on this. The principal directions for this superposed pure homogeneous deformation are arbitrarily chosen. Then, the condition that the incremental modulus associated with the third principal extension be positive is a necessary condition for material stability.

Suppose now that in the initial finite pure homogeneous deformation one pair of faces of the rectangular block is force-free and in the superposed infinitesimal pure homogeneous deformation one of the principal directions is normal to these faces, which remains force-free in the superposed deformation. We suppose also that in the superposed deformation one of the principal extension ratios, corresponding to a principal direction normal to the force-free faces, is unity. Then, the condition that the incremental modulus associated with the third principal extension be positive is a necessary condition for material stability.

2. Basic equations

We consider an isotropic elastic material to undergo a deformation in which a particle in vector position $\underset{\sim}{\xi}$ with respect to some fixed origin moves to vector position $\underset{\sim}{X}$. Let ξ_A and X_A (A=1,2,3) be the components of $\underset{\sim}{\xi}$ and $\underset{\sim}{X}$ respectively in a rectangular cartesian coordinate system x . The deformation gradient $\underset{\sim}{g}$ is defined by

$$\underset{\sim}{g} = ||g_{AB}|| = ||\partial X_A/\partial \xi_B|| \tag{2.1}$$

and the Finger strain matrix by

$$\underset{\sim}{C} = \underset{\sim}{g}\underset{\sim}{g}^{\dagger} , \tag{2.2}$$

where the dagger denotes the transpose.

The elastic properties of the material are characterized by a strain-energy function W , measured per unit initial volume. This is expressible as a function of $\underset{\sim}{C}$ through three basic strain invariants of $\underset{\sim}{C}$, denoted I_1, I_2, I_3 and defined by

$$\begin{aligned} I_1 &= \operatorname{tr} \underset{\sim}{C} , \quad I_2 = \tfrac{1}{2}\{(\operatorname{tr} \underset{\sim}{C})^2 - \operatorname{tr} \underset{\sim}{C}^2\} , \\ I_3 &= \det \underset{\sim}{C} = \tfrac{1}{6}\{I_1^3 - 3I_1 \operatorname{tr} \underset{\sim}{C}^2 + 2\operatorname{tr} \underset{\sim}{C}^3\} . \end{aligned} \tag{2.3}$$

The Cauchy stress matrix $\underset{\sim}{\sigma}$ is given by

$$\underset{\sim}{\sigma} = \frac{2}{I_3^{\frac{1}{2}}} \{(W_1 + I_1 W_2)\underset{\sim}{C} - W_2 \underset{\sim}{C}^2 + I_3 W_3 \underset{\sim}{\delta}\} , \tag{2.4}$$

where $\underset{\sim}{\delta}$ is the unit matrix and W_1, W_2, W_3 are defined by

$$W_A = \partial W/\partial I_A \quad (A=1,2,3)\ . \tag{2.5}$$

We denote by $d\underset{\sim}{g}$, $d\underset{\sim}{C}$, dI_A, $d\underset{\sim}{\sigma}$ the changes in $\underset{\sim}{g}$, $\underset{\sim}{C}$, I_A, $\underset{\sim}{\sigma}$ associated with an infinitesimal superposed deformation $\underset{\sim}{X} \to \underset{\sim}{X} + d\underset{\sim}{X}$. Then, from (2.2) and (2.3) it follows that

$$\begin{aligned} d\underset{\sim}{C} &= \underset{\sim}{g}d\underset{\sim}{g}^\dagger + (d\underset{\sim}{g})\underset{\sim}{g}^\dagger, \quad dI_1 = 2\mathrm{tr}\,\underset{\sim}{g}d\underset{\sim}{g}^\dagger\ , \\ dI_2 &= I_1 dI_1 - \mathrm{tr}(\underset{\sim}{C}d\underset{\sim}{C}),\ dI_3 = I_2 dI_1 - I_1\mathrm{tr}(\underset{\sim}{C}d\underset{\sim}{C}) + \mathrm{tr}(\underset{\sim}{C}^2 d\underset{\sim}{C}). \end{aligned} \tag{2.6}$$

From (2.4) we obtain

$$\begin{aligned} d\underset{\sim}{\sigma} = \frac{2}{I_3^{1/2}} &\{(W_1+I_1W_2)d\underset{\sim}{C} - W_2 d(\underset{\sim}{C}^2) + W_2\underset{\sim}{C}dI_1 + \tfrac{1}{2}W_3\underset{\sim}{\delta}dI_3\} \\ &- \frac{1}{I_3^{3/2}}\{(W_1+I_1W_2)\underset{\sim}{C} - W_2\underset{\sim}{C}^2\}dI_3 \\ &+ \frac{2}{I_3^{1/2}}\sum_{A=1}^{3}\{(W_{1A}+I_1W_{2A})\underset{\sim}{C} - W_{2A}\underset{\sim}{C}^2 + I_3W_{3A}\underset{\sim}{\delta}\}dI_A\ . \end{aligned} \tag{2.7}$$

3. Infinitesimal superposed pure homogeneous deformation

We now consider that the deformation $\underset{\sim}{\xi} \to \underset{\sim}{X}$ is a pure homogeneous deformation with principal extension ratios $\lambda_1, \lambda_2, \lambda_3$ and principal directions parallel to the axes of a rectangular cartesian coordinate system, $\bar{x}$ say. If n_{AB} are the components in the system $\bar{x}$ of a unit vector parallel to the axis x_A, then

$$
\begin{aligned}
& g_{AB} = \sum_P n_{AP} n_{BP} \lambda_P , \quad C_{AB} = \sum_P n_{AP} n_{BP} \lambda_P^2 , \\
& (\underset{\sim}{C}^2)_{AB} = \sum_P n_{AP} n_{BP} \lambda_P^4 , \qquad (3.1) \\
& I_1 = \lambda_1^2+\lambda_2^2+\lambda_3^2 , \; I_2 = \lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2+\lambda_1^2\lambda_2^2 , \; I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 .
\end{aligned}
$$

We also consider that the superposed infinitesimal deformation $\underset{\sim}{X} \to \underset{\sim}{X} + d\underset{\sim}{X}$ is a pure homogeneous deformation with principal extension ratios $1 + d\lambda_1$, $1 + d\lambda_2$, $1 + d\lambda_3$ and principal directions parallel to the axes of the system x. Then, from (2.6),

$$
\begin{aligned}
& dg_{AB} = g_{AB} d\lambda_A , \; dC_{AB} = C_{AB}(d\lambda_A + d\lambda_B), \; dI_1 = 2 \sum_A C_{AA} d\lambda_A , \\
& dI_2 = 2 \sum_A \{I_1 C_{AA} - (\underset{\sim}{C}^2)_{AA}\} d\lambda_A , \\
& dI_3 = 2 \sum_A (\underset{\sim}{C}^3 - I_1 \underset{\sim}{C}^2 + I_2 \underset{\sim}{C})_{AA} d\lambda_A = 2 I_3 \sum_A d\lambda_A , \qquad (3.2) \\
& d(\underset{\sim}{C}^2)_{AB} = 2(\underset{\sim}{C}^2)_{AB}(d\lambda_A + d\lambda_B) + 2 \sum_P C_{AP} C_{PB} d\lambda_P .
\end{aligned}
$$

4. Constrained simple extension

We now envisage that, after the material is subjected to the pure homogeneous deformation with extension ratios $\lambda_1,\lambda_2,\lambda_3$ and principal directions parallel to the axes of the system $\bar{x}$, a rectangular block with its edges parallel to the axes of the system x is cut from it and held in its deformed state by appropriate forces applied to its faces. It is then subjected to an infinitesimal pure homogeneous deformation, with principal extension ratio $(1+d\lambda_1)$ in the x_1-direction, while its dimensions parallel to the x_2 and x_3-directions are held constant.

In this infinitesimal deformation, only the normal forces acting on the faces perpendicular to the x_1-direction do work. We accordingly calculate the incremental modulus μ , say, for this extension parallel to the x_1-direction. It is given by

$$\mu = \left(\frac{\partial\sigma_{11}}{\partial\lambda_1}\right)_{\lambda_2,\lambda_3} . \qquad (4.1)$$

With (3.1) and

$$d\lambda_A = (d\lambda_1,0,0) , \qquad (4.2)$$

equations (3.2) yield

$$\begin{aligned} dg_{11} = g_{11}d\lambda_1,\ dC_{11} = 2C_{11}d\lambda_1,\ dI_1 = 2C_{11}d\lambda_1, \\ dI_2 = 2\{I_1C_{11}-(\underset{\sim}{C}^2)_{11}\}d\lambda_1 , \qquad (4.3) \\ dI_3 = 2I_3d\lambda_1,\ d(\underset{\sim}{C}^2)_{11} = 2\{(\underset{\sim}{C}^2)_{11} + C_{11}^2\}d\lambda_1 . \end{aligned}$$

Then, (2.7) and (4.1) yield

$$\mu = \frac{2}{I_3^{1/2}} \{L_1 W + 2L_1^2 W\} , \tag{4.4}$$

where L_1 is the linear operator defined by

$$L_1 = C_{11} \frac{\partial}{\partial I_1} + [I_1 C_{11} - (\underset{\sim}{C}^2)_{11}] \frac{\partial}{\partial I_2} + I_3 \frac{\partial}{\partial I_3} . \tag{4.5}$$

With (3.1), the expression (4.5) may be written as

$$\begin{aligned} L_1 = {} & (n_{11}^2\lambda_1^2 + n_{12}^2\lambda_2^2 + n_{13}^2\lambda_3^2) \frac{\partial}{\partial I_1} + [\lambda_1^2(\lambda_2^2+\lambda_3^2)n_{11}^2 \\ & + \lambda_2^2(\lambda_3^2+\lambda_1^2)n_{12}^2 + \lambda_3^2(\lambda_1^2+\lambda_2^2)n_{13}^2] \frac{\partial}{\partial I_2} + I_3 \frac{\partial}{\partial I_3} . \end{aligned} \tag{4.6}$$

In the Appendix, §7, we have derived the conditions that μ have a positive minimum value for real values of n_{11}, n_{12}, n_{13}. With the notation

$$\begin{aligned} M_A &= 2(W_{11} + 2\lambda_A^2 W_{12} + \lambda_A^4 W_{22}) , \quad \Delta = W_{11}W_{22} - W_{12}^2 , \\ \Theta &= W_1 W_{12} - W_2 W_{11} + 4I_3(W_{13}W_{12} - W_{23}W_{11}) , \\ \Phi &= W_1 W_{22} - W_2 W_{12} + 4I_3(W_{13}W_{22} - W_{23}W_{12}) , \end{aligned} \tag{4.7}$$

these conditions are

$$\begin{aligned} & M_A > 0 , \quad \Delta > 0 , \\ & \frac{1}{2\lambda_A^2 I_1 - I_2 - 3\lambda_A^4} \left(\Theta + \lambda_A^2 \Phi - \frac{\Delta I_3}{\lambda_A^2}\right) \geq 0 , \end{aligned} \tag{4.8}$$

and

$$2\Delta\left(-\Phi \frac{\partial}{\partial I_1} + \Theta \frac{\partial}{\partial I_2} + 4\Delta I_3 \frac{\partial}{\partial I_3}\right)W$$
$$+ 2\left(-\Phi \frac{\partial}{\partial I_1} + \Theta \frac{\partial}{\partial I_2} + 4\Delta I_3 \frac{\partial}{\partial I_3}\right)^2 W > 0 \ . \tag{4.9}$$

If any of the conditions (4.8) is not satisfied then μ does not possess a minimum stationary value for any real value of n_{1A}. In this case μ has its least value when the vector n_{1A} lies in a principal plane of the underlying pure homogeneous deformation. If, on the other hand, the conditons (4.8) are satisfied, but (4.9) is not, the minimum value of μ is negative, i.e. the necessary condition for material stability is violated.

5. Constrained simple extension in a principal plane

We shall now consider the special case in which the x_3 and $\bar{x}_3$ axes coincide, so that the direction of the infinitesimal extension lies in a principal plane of the underlying finite pure homogeneous deformation, the $\bar{x}_1\bar{x}_2$ plane. We then have $n_{13} = 0$, and the expression (4.6) for L_1 becomes

$$L_1 = (n_{11}^2\lambda_1^2+n_{12}^2\lambda_2^2)\frac{\partial}{\partial I_1} + [\lambda_1^2(\lambda_2^2+\lambda_3^2)n_{11}^2 + \lambda_2^2(\lambda_3^2+\lambda_1^2)n_{12}^2]\frac{\partial}{\partial I_2} + I_3\frac{\partial}{\partial I_3}\,. \tag{5.1}$$

With (5.1) the expression (4.4) for μ may be written as

$$\begin{aligned}\mu = \frac{2}{I_3^{\frac{1}{2}}}\{&n_{11}^4[\lambda_1^2(K_3+\lambda_2^2L_3) + c_{11}] \\ &+ n_{11}^2n_{12}^2[\lambda_1^2(K_3+\lambda_2^2L_3) + \lambda_2^2(K_3+\lambda_1^2L_3) + 2c_{12}] \\ &+ n_{12}^4[\lambda_2^2(K_3+\lambda_1^2L_3) + c_{22}]\}\,,\end{aligned} \tag{5.2}$$

where

$$K_A = W_1+\lambda_A^2W_2\,, \qquad L_A = W_2+\lambda_A^2W_3\,,$$

$$\begin{aligned}\tfrac{1}{2}c_{AB} = \lambda_A^2\lambda_B^2\,&\{W_{11}+(2I_1-\lambda_A^2-\lambda_B^2)W_{12} + (I_1-\lambda_A^2)(I_1-\lambda_B^2)W_{22}\} \\ &+ I_3\{(\lambda_A^2+\lambda_B^2)W_{31} + [\lambda_A^2(I_1-\lambda_A^2) + \lambda_B^2(I_1-\lambda_B^2)]W_{32} \\ &+ I_3W_{33}\}\,.\end{aligned} \tag{5.3}$$

We can easily find the necessary and sufficient conditions that μ, given by (5.2), be positive for all choices of n_{11}, n_{12}. By taking $n_{11}, n_{12} = 1, 0$ and $0, 1$, we see that the conditions

$$F_{13} > 0 \quad \text{and} \quad F_{23} > 0 \,, \tag{5.4}$$

with F_{13}, F_{23} defined by

$$F_{13} = K_3 + \lambda_2^2 L_3 + c_{11}/\lambda_1^2 \,, \quad F_{23} = K_3 + \lambda_1^2 L_3 + c_{22}/\lambda_2^2 \,, \tag{5.5}$$

are necessary. If these conditions are satisfied, we can rewrite (5.2) as

$$\begin{aligned} \tfrac{1}{2} I_3^{\frac{1}{2}} \mu &= (n_{11}^2 \lambda_1 F_{13}^{\frac{1}{2}} - n_{12}^2 \lambda_2 F_{23}^{\frac{1}{2}})^2 \\ &\quad + n_{11}^2 n_{12}^2 [(\lambda_1 F_{13}^{\frac{1}{2}} + \lambda_2 F_{23}^{\frac{1}{2}})^2 + 2c_{12} - c_{11} - c_{22}] \,. \end{aligned} \tag{5.6}$$

Accordingly the necessary and sufficient conditions that μ be positive for all choices of n_{11}, n_{12} are

$$2\lambda_1 \lambda_2 F_{13}^{\frac{1}{2}} F_{23}^{\frac{1}{2}} + \lambda_1^2 (K_3 + \lambda_2^2 L_3) + \lambda_2^2 (K_3 + \lambda_1^2 L_3) + 2c_{12} > 0 \,, \tag{5.7}$$

together with the conditions (5.4).

Of course, if μ is to be positive for all choices of n_{1A} parallel to any of the principal planes of the underlying pure homogeneous deformation, the corresponding conditions are

$$\begin{aligned} &F_{AC} = K_C + \lambda_B^2 L_C + \frac{c_{AA}}{\lambda_A^2} > 0 \,, \quad F_{BC} = K_C + \lambda_A^2 L_C + \frac{c_{BB}}{\lambda_B^2} > 0 \,, \\ &2\lambda_A \lambda_B F_{AC}^{\frac{1}{2}} F_{BC}^{\frac{1}{2}} + \lambda_A^2 (K_C + \lambda_B^2 L_C) + \lambda_B^2 (K_C + \lambda_A^2 L_C) + 2c_{AB} > 0 \end{aligned} \tag{5.8}$$

$$(ABC = 123, 231, 312),$$

where K_A, L_A and c_{AB} are defined in (5.3).

6. Extension with force-free surfaces

From (3.2) and (2.7) we can obtain an expression for the change $d\sigma_{11}$ in the normal component of the Cauchy stress, in the x_1-direction, associated with an infinitesimal pure homogeneous deformation with principal extension ratios $1 + d\lambda_1$, $1 + d\lambda_2$, $1 + d\lambda_3$ and principal directions parallel to the axes of the coordinate system x:

$$
\begin{aligned}
d\sigma_{11} = \frac{2}{I_3^{\frac{1}{2}}} [& \{2C_{11}W_1 + 2[I_1C_{11} - (\underset{\sim}{C}^2)_{11}]W_2\}d\lambda_1 \\
& + \sum_A \{-C_{11}W_1 - [I_1C_{11} - (\underset{\sim}{C}^2)_{11}]W_2 + I_3W_3\}d\lambda_A \\
& + 2\{C_{11} \sum_A C_{AA}d\lambda_A - \sum_A C_{1A}C_{A1}d\lambda_A\}W_2 \\
& + 2\{\sum_A (L_1L_AW)d\lambda_A\}] ,
\end{aligned} \tag{6.1}
$$

where L_A is the linear operator defined by (cf.(4.5))

$$
L_A = C_{AA}\frac{\partial}{\partial I_1} + [I_1C_{AA} - (\underset{\sim}{C}^2)_{AA}]\frac{\partial}{\partial I_2} + I_3\frac{\partial}{\partial I_3} . \tag{6.2}
$$

Let $d\pi_{11}$ be the normal component of the Piola-Kirchhoff stress in the x_1-direction, associated with the infinitesimal pure homogeneous deformation, and based on the configuration prior to this deformation as the reference configuration. Then,

$$
d\pi_{11} = d\sigma_{11} + \sigma_{11}(d\lambda_2 + d\lambda_3) . \tag{6.3}
$$

Introducing the expressions (2.4) and (6.1) for σ_{11} and $d\sigma_{11}$, we obtain, with (6.2),

$$d\pi_{11} = \frac{2}{I_3^{\frac{1}{2}}} \{L_1 W d\lambda_1 + 2I_3 W_3 (d\lambda_2 + d\lambda_3)$$
$$+ 2(C_{11} \sum_A C_{AA} - \sum_A C_{1A} C_{A1}) W_2 d\lambda_A$$
$$+ 2 \sum_A (L_1 L_A W) d\lambda_A\} \,. \tag{6.4}$$

Corresponding expressions for $d\pi_{22}$ and $d\pi_{33}$ can be written down by appropriate permutation of the subscripts.

We now suppose that in the initial finite pure homogeneous deformation, the normal stress in the x_3-direction is zero. We choose the x_3-axis of the coordinate system x to coincide with the $\bar{x}_3$-axis. Furthermore, it is assumed that in the superposed pure homogeneous deformation the faces of the block normal to the x_3-axis remain force-free, while the x_2-dimension of the block is held constant. We shall call this superposed deformation <u>partially constrained simple extension</u>. Then, the only forces which do work in the superposed pure homogeneous deformation are those applied normally to the faces of the block which are normal to the x_1-direction. We shall calculate the incremental modulus μ in this case. μ is defined by

$$\mu = d\pi_{11}/d\lambda_1 \,. \tag{6.5}$$

The condition that the faces of the block normal to the x_3-direction are force-free yields

$$\sigma_{33} = 0 \quad \text{and} \quad d\pi_{33} = 0 \,. \tag{6.6}$$

We now introduce the condition that the dimension of the block parallel to the x_2-axis is maintained constant in the infinitesimal superposed deformation, i.e. $d\lambda_2 = 0$. Then, bearing in mind that $C_{13} = 0$, (6.4) yields

$$d\pi_{11} = \frac{2}{I_3^{\frac{1}{2}}} \{[(L_1+2L_1^2)W]d\lambda_1 + 2[C_{11}\lambda_3^2W_2+I_3W_3+L_1L_3W]d\lambda_3\}. \qquad (6.7)$$

From the expression for $d\pi_{33}$ corresponding to (6.7), we obtain the condition $(6.6)_2$ for the surfaces normal to the x_3-axis to be force-free as

$$[(L_3+2L_3^2)W]d\lambda_3 + 2(C_{11}\lambda_3^2W_2+I_3W_3+L_1L_3W)d\lambda_1 = 0 . \qquad (6.8)$$

The condition $(6.6)_1$ yields, with (2.4),

$$L_3W = W_1C_{33} + W_2[C_{33}(C_{11}+C_{22})-(C_{13}^2+C_{23}^2)] + I_3W_3 = 0 . \qquad (6.9)$$

From (6.7) and (6.8) we obtain, with (6.9),

$$\mu = 2I_3^{-\frac{1}{2}}(L_3^2W)^{-1}\{(L_1+2L_1^2)WL_3^2W-2(C_{11}\lambda_3^2W_2+I_3W_3+L_1L_3W)^2\} . \qquad (6.10)$$

We note from (4.4) and (6.9) that

$$L_3^2W > 0 \qquad (6.11)$$

is the necessary and sufficient condition that the incremental modulus for constrained simple extension in the x_3-direction be positive, if the faces normal to the x_3-direction are force-free in the underlying pure homogeneous deformation. With the condition (6.11) it follows from (6.10) that the necessary and sufficient condition for μ to be positive is

$$(L_1+2L_1^2)WL_3^2W > 2(C_{11}\lambda_3^2W_2+I_3W_3+L_1L_3W)^2 \ . \tag{6.12}$$

We see, as in §§4 and 5, that (cf. equations (4.4) and (5.2))

$$(L_1+2L_1^2)W = \lambda_1^2F_{13}n_{11}^4+(\lambda_1^2F_{13}+\lambda_2^2F_{23}+2c_{12}-c_{11}-c_{22})n_{11}^2n_{12}^2 + \lambda_2^2F_{23}n_{12}^4 \ , \tag{6.13}$$

where F_{13}, F_{23} and c_{AB} are defined in (5.3) and (5.5). Also, with $(3.1)_2$ and $n_{3A} = (0,0,1)$ equation (6.2) yields

$$L_1 = (\lambda_1^2n_{11}^2+\lambda_2^2n_{12}^2)\frac{\partial}{\partial I_1} + [\lambda_1^2(\lambda_2^2+\lambda_3^2)n_{11}^2+\lambda_2^2(\lambda_3^2+\lambda_1^2)n_{12}^2]\frac{\partial}{\partial I_2} + I_3\frac{\partial}{\partial I_3} \ , \tag{6.14}$$

$$L_3 = \lambda_3^2\left[\frac{\partial}{\partial I_1} + (\lambda_1^2+\lambda_2^2)\frac{\partial}{\partial I_2} + \lambda_1^2\lambda_2^2\frac{\partial}{\partial I_3}\right] .$$

Thus,

$$\begin{aligned} L_1L_3W &= \tfrac{1}{2}(c_{13}n_{11}^2+c_{23}n_{12}^2) \ , \\ L_3^2W &= \tfrac{1}{2}\,c_{33} \ . \end{aligned} \tag{6.15}$$

With (6.13), (6.14) and (6.15), the relation (6.12) can be rewritten as

$$\alpha n_{11}^4 + 2\beta n_{11}^2 n_{12}^2 + \gamma n_{12}^4 > 0 \ , \tag{6.16}$$

where

$$\begin{aligned}
\alpha &= \lambda_1^2 F_{13} c_{33} - 4(\lambda_1^2\lambda_3^2 L_2 + \tfrac{1}{2}c_{13})^2 \ , \\
\gamma &= \lambda_2^2 F_{23} c_{33} - 4(\lambda_2^2\lambda_3^2 L_1 + \tfrac{1}{2}c_{23})^2 \ , \\
\beta &= \tfrac{1}{2}\, c_{33}(\lambda_1^2 F_{13} + \lambda_2^2 F_{23} + 2c_{12} - c_{11} - c_{22}) \\
&\qquad - 4(\lambda_1^2\lambda_3^2 L_2 + \tfrac{1}{2}c_{13})(\lambda_2^2\lambda_3^2 L_1 + \tfrac{1}{2}c_{23}) \ .
\end{aligned} \tag{6.17}$$

By means of an argument similar to that used in §5, we see that the necessary and sufficient conditions that (6.16) be satisfied for all choices of n_{11}, n_{12} are

$$\alpha > 0 \ , \quad \gamma > 0 \ , \quad \beta + \alpha^{\frac{1}{2}}\gamma^{\frac{1}{2}} > 0 \ . \tag{6.18}$$

These are the necessary and sufficient conditions that the incremental modulus be positive for partially constrained extensions parallel to the principal plane $\bar{x}_1\bar{x}_2$ of the underlying pure homogeneous deformation. The corresponding conditions for partially constrained extensions parallel to the other two principal planes can be obtained by cyclic permutation of the subscripts in the expressions (6.17) for α,β,γ .

7. Appendix

In order to determine the values of n_{1A} for which μ, given by (4.4), has a stationary value, it is convenient to introduce the notation $N_A = n_{1A}^2$. Then, substituting for N_3 from

$$N_1 + N_2 + N_3 = 1 \tag{7.1}$$

in (4.6) and introducing the resulting expression for L_1 in (4.4), we obtain

$$\mu = \frac{2}{I_3^{\frac{1}{2}}} (\alpha_{11}N_1^2 + 2\alpha_{12}N_1N_2 + \alpha_{22}N_2^2 + 2\beta_1N_1 + 2\beta_2N_2 + \gamma) , \tag{7.2}$$

where

$$\alpha_{11} = 2M_1^2W , \quad \alpha_{22} = 2M_2^2W , \quad \alpha_{12} = 2M_1M_2W ,$$

$$2\beta_1 = (M_1 + 4M_1N)W , \quad 2\beta_2 = (M_2 + 4M_2N)W ,$$

$$\gamma = (N + 2N^2)W , \tag{7.3}$$

and M_1, M_2, N are linear operators defined by

$$M_1 = (\lambda_1^2 - \lambda_3^2)(\frac{\partial}{\partial I_1} + \lambda_2^2 \frac{\partial}{\partial I_2}) , \quad M_2 = (\lambda_2^2 - \lambda_3^2)(\frac{\partial}{\partial I_1} + \lambda_1^2 \frac{\partial}{\partial I_2}) ,$$

$$N = \lambda_3^2[\frac{\partial}{\partial I_1} + (\lambda_1^2 + \lambda_2^2)\frac{\partial}{\partial I_2} + \lambda_1^2\lambda_2^2 \frac{\partial}{\partial I_3}] . \tag{7.4}$$

μ has a stationary value when N_1, N_2 satisfy the equations

$$\alpha_{11}N_1 + \alpha_{12}N_2 + \beta_1 = 0 , \quad \alpha_{12}N_1 + \alpha_{22}N_2 + \beta_2 = 0 . \tag{7.5}$$

These equations yield

$$N_1 = \frac{\alpha_{12}\beta_2-\alpha_{22}\beta_1}{\alpha_{11}\alpha_{22}-\alpha_{12}^2}\ , \quad N_2 = \frac{\alpha_{12}\beta_1-\alpha_{11}\beta_2}{\alpha_{11}\alpha_{22}-\alpha_{12}^2}\ . \tag{7.6}$$

These values of N_1, N_2 yield real values of n_{1A} provided that

$$N_1 \geq 0\ , \quad N_2 \geq 0 \quad \text{and} \quad N_1 + N_2 \leq 1\ . \tag{7.7}$$

Since the values of N_1, N_2 for which μ has a stationary value satisfy equations (7.5), it follows from (7.2) that this stationary value of μ, if it exists, is given by

$$\mu = \frac{2}{I_3^{\frac{1}{2}}}(N_1\beta_1+N_2\beta_2+\gamma)\ . \tag{7.8}$$

It is easily seen, by considering the second variation of μ, that the values of N_1, N_2 given by (7.6) yield a minimum stationary value for μ if and only if

$$\alpha_{11} > 0\ , \ \alpha_{22} > 0\ , \text{ and } \alpha_{11}\alpha_{22}-\alpha_{12}^2 > 0\ . \tag{7.9}$$

With (7.3) and (4.7), we note that the conditions (7.9) are in fact $(4.8)_1$, with $A=1,2$, and $(4.8)_2$. In turn, these conditions imply that $M_3 > 0$. To see this, we note the identity

$$16\Lambda^2\Delta + [(\lambda_2^2-\lambda_3^2)^2M_1 - (\lambda_1^2-\lambda_3^2)^2M_2]^2 + (\lambda_1^2-\lambda_2^2)^4M_2^2$$
$$= 2(\lambda_1^2-\lambda_2^2)^2M_3\{(\lambda_2^2-\lambda_3^2)^2M_1 + (\lambda_1^2-\lambda_3^2)^2M_2\}\ , \tag{7.10}$$

where

$$\Lambda = (\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2) \ . \tag{7.11}$$

Then, since $\Delta > 0$, the left-hand member of (7.10) is positive and, since $M_1 > 0$, $M_2 > 0$, we conclude that $M_3 > 0$.

Using the expressions (7.3) for the α's and β's in (7.6), we can obtain expressions for N_1, N_2, and from (7.1) we obtain N_3. Thus, with the notation (4.7) and (7.11), we obtain

$$4\Lambda\Delta N_1 = (\lambda_2^2-\lambda_3^2)(\Theta+\lambda_1^2\Phi-4\lambda_2^2\lambda_3^2\Delta) \ , \tag{7.12}$$

while corresponding expressions for N_2 and N_3 are obtained by cyclic permutation of the indices on the λ's. We use these expressions for N_1, N_2, N_3 to obtain

$$\begin{aligned} &4\Delta(\lambda_1^2N_1+\lambda_2^2N_2+\lambda_3^2N_3) = -\Phi \ , \\ &4\Delta[\lambda_1^2(\lambda_2^2+\lambda_3^2)N_1 + \lambda_2^2(\lambda_3^2+\lambda_1^2)N_2 + \lambda_3^2(\lambda_1^2+\lambda_2^2)N_3] = \Theta \ . \end{aligned} \tag{7.13}$$

From (7.13), (4.4) and (4.6), we have

$$\begin{aligned} \mu = \frac{1}{8\Delta^2 I_3^{\frac{1}{2}}} \{ &4\Delta[-\Phi \frac{\partial}{\partial I_1} + \Theta \frac{\partial}{\partial I_2} + 4\Delta I_3 \frac{\partial}{\partial I_3}]W \\ &+ 2[-\Phi \frac{\partial}{\partial I_1} + \Theta \frac{\partial}{\partial I_2} + 4\Delta I_3 \frac{\partial}{\partial I_3}]^2 W\} \ . \end{aligned} \tag{7.14}$$

From (7.12), with (7.11) and $(4.8)_2$, if follows that the N's are all non-negative if

$$\frac{1}{(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)} \quad (\Theta+\lambda_1^2\Phi - 4\lambda_2^2\lambda_3^2\Delta) \geq 0 \tag{7.15}$$

and the two conditions obtained from it by cyclic permutation of the subscripts on the λ's are satisfied.

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Two sets of restrictions on the strain-energy function for a compressible isotropic elastic material are obtained which are necessary conditions for stability of the material. These arise from the following considerations. (i) A rectangular block is subjected to a finite pure homogeneous deformation and an infinitesimal pure homogeneous deformation with arbitrary principal directions is superposed. The dimensions in two of these principal directions are held constant. Then the incremental modulus associated with the third

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) principal direction must be positive for stability to obtain. (ii) In the initial pure homogeneous deformation one pair of faces of the block is force-free. The superposed infinitesimal pure homogeneous deformation has one of its principal directions normal to these faces, which remain force-free, and the principal extension ratio corresponding to another is unity. The incremental modulus corresponding to the third principal direction must be positive for stability to obtain.

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